PROPAGATION OF WEAK SHOCK WAVES IN A MAGNETIC FIELD

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 4, pp. 91–93, 1966

This paper gives a solution of the problem of the propagation of weak shock waves in an inhomogeneous conducting medium in the presence of a magnetic field. The width of the perturbed region is taken to be small compared with the characteristic dimensions of the problem. The magnetic Reynolds number is also assumed small, which allows one to neglect the induced magnetic field. The method of solution employed is similar to that used in [1–3].

§1. We shall consider a medium with isotropic conductivity. The magnetic Reynolds number is assumed small ($R_m \ll 1$), and we neglect the induced magnetic field which arises when a conducting medium moves in an external magnetic field. The viscosity and thermal conductivity are also neglected. With these assumptions the system of equations of magnetogasdynamics has the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_k}\rho v_k=0, \qquad \rho\frac{\partial v_i}{\partial t}+\rho v_k\frac{\partial v_i}{\partial x_k}+\frac{\partial p}{\partial x_i}=-F_i,$$

$$\frac{\partial p}{\partial t}+v_k\frac{\partial p}{\partial x_k}-a^2\left(\frac{\partial\rho}{\partial t}+v_k\frac{\partial\rho}{\partial x_k}\right)=(\gamma-1)\frac{j^2}{\sigma},$$

$$\mathbf{F}=\frac{1}{c}\mathbf{H}\times\mathbf{j}, \quad \mathbf{j}=\frac{\sigma}{c}\mathbf{v}\times\mathbf{H}, \quad \gamma p=a^2\rho. \tag{1.1}$$

Here v_i, ρ, p, a are the velocity, density, pressure, and velocity of sound, respectively; t is time; x_i are Cartesian coordinates; the indices i, k take the values 1, 2, 3; H is the magnetic field; j is the current density, σ is the conductivity of the medium, and c is the velocity of light.

Let $\varphi(t, x_i)$ = const be the characteristic surfaces of the system (1.1).

The equation for $\varphi(t, x_i)$ will have the form

$$\left[\left(\frac{\partial\varphi}{\partial t}+v_k\frac{\partial\varphi}{\partial x_k}\right)^2-a^2\left(\frac{\partial\varphi}{\partial x_k}\right)^2\right]\left(\frac{\partial\varphi}{\partial t}+v_k\frac{\partial\varphi}{\partial x_k}\right)^3=0, \tag{1.2}$$

from which it is clear that the equation

$$\frac{\partial\varphi}{\partial t}+v_k\frac{\partial\varphi}{\partial x_k}\pm a\left[\left(\frac{\partial\varphi}{\partial x_k}\right)^2\right]^{1/2}=0 \tag{1.3}$$

defines two families of characteristics C_+ and C_-, and the equation

$$\frac{\partial\varphi}{\partial t}+v_k\frac{\partial\varphi}{\partial x_k}=0 \tag{1.4}$$

the family of characteristics C_0.

Following [1], we introduce the surfaces N_+, N_- and S moving in space (x_i), instead of the characteristic surfaces C_+, C_- and C_0, which are stationary in space (t, x_i). According to (1.3) the velocity of propagation of the surfaces N_+ and N_- is equal to $v_i \pm a n_i$, where n_i is the normal to the appropriate surface N_+ or N_-:

$$n_i=\frac{\partial\varphi/\partial x_i}{\sqrt{(\partial\varphi/\partial x_k)^2}}.$$

The surfaces S, corresponding to the characteristic surfaces C_0, are fluid surfaces moving together with the particles of the medium.

Reducing the system (1.1) to characteristic form, we obtain

$$(v_k\pm an_k)\left(\frac{\partial p}{\partial x_k}\right)\pm\rho a(n_iv_k\pm a\delta_{ik})\left(\frac{\partial v_i}{\partial x_k}\right)=$$

$$=\mp an_kF_k+(\gamma-1)\frac{j^2}{\sigma}, \tag{1.5}$$

$$\rho s_iv_k\left(\frac{\partial v_i}{\partial x_k}\right)+s_i\left(\frac{\partial p}{\partial x_i}\right)=-s_iF_i,$$

$$v_k\left(\frac{\partial p}{\partial x_k}\right)-a^2v_k\left(\frac{\partial\rho}{\partial x_k}\right)=(\gamma-1)\frac{j^2}{\sigma}. \tag{1.6}$$

Here the operator

$$\left(\frac{\partial}{\partial x_k}\right)=\frac{\partial}{\partial x_k}-\frac{\partial\varphi/\partial x_k}{\partial\varphi/\partial t}\frac{\partial}{\partial t} \tag{1.7}$$

denotes the derivative along the characteristic surface; the upper sign in (1.5) corresponds to the C_+ characteristics, and the lower to those of C_-; the vector s_i in (1.6) satisfies the condition $s_i\,\partial\varphi/\partial x_i = 0$. Equations (1.6) and (1.7) correspond to the characteristics of C_0.

§2. Using the characteristic system (1.5)–(1.7), obtained above, we shall consider the propagation of weak shock waves in a medium for which the pressure p_0, the density ρ_0, velocity U_i, magnetic field H_i and velocity of sound a_0 are given as functions of the coordinates (but are independent of time). We shall consider the perturbed quantities

$$\Delta=p-p_0, \quad \delta=\rho-\rho_0, \quad u_i=v_i-U_i$$

to be small compared with p_0, ρ_0 and a_0 respectively. We shall take the width of the perturbed region λ to be small compared with the radius of curvature of the shock front R and the characteristic length L within which the medium and magnetic field change substantially. In addition, we shall assume that Δ, δ and u_i change by an amount of the order of themselves over distances of the order of R, L in directions tangential to the front. By hypothesis the magnetic Reynolds number $R_m \ll 1$, and we have

$$R_m=4\pi\sigma\lambda a_0/c^2\ll 1$$

since λ and a_0 are the characteristic dimension and velocity in the problem under consideration.

Setting Δ, δ and u_i in the characteristic equations corresponding to the C_0 and C_- characteristics, and integrating them along trajectories of elements of the surfaces S and N_-, respectively, we find, as in

[2], that the following relations are obeyed in the perturbed region (quantities of the order of Δ^2, $\lambda\Delta/R$, $\lambda\Delta/L$, R_m compared with Δ are neglected):

$$u_i = un_i, \quad \Delta = a_0^2\delta, \quad \Delta = \rho_0 a_0 \; u \; (u = \sqrt{u_k^2}). \qquad (2.1)$$

Under the assumptions made above, the relations (2.1) coincide with the relations at the shock front [4], if terms of the order of Δ^2, R, $\lambda\Delta/R$, $\lambda\Delta/L$, $R_m\Delta$ are neglected in the latter relations, and consequently, to the degree of accuracy indicated, the shock front does not influence the flow behind it. Relations (2.1) coincide with those obtained in [2], i.e., the interaction of the conducting medium and magnetic field does not affect the relations between the parameters of the medium in the perturbed region (under the restrictions which have been imposed).

Setting Δ, δ and u_i in Eq. (1.5), corresponding to the C_+ characteristics, and taking (2.1) into account, we obtain

$$2\frac{d\Delta}{dt} - \frac{\Delta}{\rho_0 a_0}\frac{d\rho_0 a_0}{dt} + \Delta\left[a_0\frac{\partial n_k}{\partial x_k} + \gamma\frac{\partial U_k}{\partial x_k} + n_i n_k\frac{\partial U_i}{\partial x_k} + \frac{\gamma-1}{2\rho_0 a_0}\frac{\sigma}{c^2}(\mathbf{n}\cdot[\mathbf{H}\times[\mathbf{U}\times\mathbf{H}]]) + \frac{\sigma}{\rho_0 c^2}(\mathbf{n}\cdot[\mathbf{H}\times[\mathbf{n}\times\mathbf{H}]])\right] =$$
$$= \rho_0 a_0(n_i n_k + a_0\delta_{ik})\frac{\partial}{\partial x_k}\left(\frac{\Delta}{\rho_0 a_0}n_i - u_i\right) + \left(\frac{a_0\delta}{\rho_0}n_k - u_k\right)\frac{\partial p_0}{\partial x_k} +$$
$$+ n_i(\Delta n_k - \rho_0 a_0 u_k)\frac{\partial U_i}{\partial x_k} - \frac{(\gamma-1)}{c^2\rho_0 a_0}\sigma(\mathbf{n}\cdot[\mathbf{H}\times[\mathbf{u}\times\mathbf{H}]]) + (\gamma-1)\frac{j^2}{\sigma}, \qquad (2.2)$$

$$\frac{d}{dt} = (v_k + a_0 n_k)\frac{\partial}{\partial x_k}. \qquad (2.3)$$

Here the operator (2.3) denotes the derivative along the ray—trajectory of motion of the element of surface N_+. Integrating this equation along the ray and neglecting small terms of order Δ^2, $\lambda\Delta/R$, $\lambda\Delta/L$, $R_m\Delta$ compared with Δ, we obtain

$$\Delta = \frac{\alpha\sqrt{\rho_0 a_0}}{L},$$

$$L = \exp\left\{\frac{1}{2}\int_0^t\left[a_0\frac{\partial n_k}{\partial x_k} + \gamma\frac{\partial U_k}{\partial x_k} + n_i n_k\frac{\partial U_i}{\partial x_k} - \frac{3}{2}\frac{(\gamma-1)\sigma}{a_0\rho_0 c^2}([\mathbf{n}\times\mathbf{H}]\cdot[\mathbf{U}\times\mathbf{H}]) + \frac{\sigma}{\rho_0 c^2}([\mathbf{n}\times\mathbf{H}])^2\right]dt\right\}. \qquad (2.4)$$

The quantity α (on a given ray) depends on the number of the surface N_+. The ray equations are obtained from (1.3) and have the form

$$\frac{dx_i}{dt} = U_i + a_0 n_i, \qquad \frac{dn_i}{dt} = (n_i n_k - \delta_{ik})\left(\frac{\partial a_0}{\partial x_k} + n_j\frac{\partial U_j}{\partial x_k}\right). \qquad (2.5)$$

Repeating the procedure carried out in [2], we obtain the equation for determining the change of α at the shock front along the ray:

$$\alpha^2\int_0^t\frac{dt}{(a_0 + U_k n_k)\sqrt{\rho_0 a_0}\,L} = \frac{4}{\gamma+1}\int_\alpha^{\alpha_0}\alpha f'(\alpha)\,d\alpha, \qquad (2.6)$$

where $f(\alpha)$ is an arbitrary function determined by the pressure distribution in the wave given for the time $t = 0$, α_0 is the value at the shock front for $t = 0$.

The pressure at the front of a shock wave with a linear profile behind the front varies in virtue of (2.4), (2.6), according to the law

$$\Delta = \frac{\alpha_0\sqrt{\rho_0 a_0}}{L}\left(1 + \frac{\gamma+1}{2}\frac{\alpha_0(a_{00} + U_{0k}n_{0k})}{\lambda_0}\times\int_0^t\frac{dt}{(a_0 + U_k n_k)\sqrt{\rho_0 a_0}\,L}\right)^{-1/2}. \qquad (2.7)$$

Here a_{00}, U_{0k} and n_{0k} are the values of a_0, U_{0k} and n_k at the front at the time $t = 0$; λ_0 is the width of the region $\Delta > 0$ for $t = 0$.

The form of expression (2.6) is the same as that obtained in [2], but the presence of additional magnetic terms in the exponent of (2.3) alters the picture of shock wave damping radically.

In the case of a homogeneous stationary medium and uniform magnetic field in a direction parallel to the shock front, we have from (2.6) for a one-dimensional plane shock wave and cylindrical shock wave, respectively,

$$\Delta = \alpha_0\sqrt{\rho_0 a_0}e^{-Bt}\left[1 + \frac{\gamma+1}{2}\frac{\alpha_0}{\lambda_0}\frac{1}{\sqrt{\rho_0 a_0}B}(1 - e^{-Bt})\right]^{-1/2}, \qquad (2.8)$$

$$\Delta = \alpha_0\sqrt{\rho_0 a_0}\frac{e^{-Bt}}{\sqrt{1 + a_0 t/r_0}}\left[1 + \frac{\gamma+1}{2}\frac{\alpha_0}{\lambda_0}\frac{2}{\sqrt{\rho_0 a_0}}\times\left(\frac{r_0}{a_0 B}\right)^{1/2}e^{Br_0/a_0}\left(\Phi(w) - \Phi\left(\left(\frac{Br_0}{a_0}\right)^{1/2}\right)\right)\right]^{-1/2}, \qquad (2.9)$$

$$B = \frac{\sigma H^2}{2\rho_0 c^2}, \qquad \Phi(w) = \frac{2}{\sqrt{\pi}}\int_0^w e^{-z^2}dz, \qquad w = \left(\frac{Br_0}{a_0}\right)^{1/2}\left(1 + \frac{a_0}{r_0}t\right)^{1/2}.$$

Here r_0 is the position of the front for $t = 0$. The one-dimensional spherical case does not exist for a uniform magnetic field, since the magnetic field does not exert any influence on the damping of shock waves along rays parallel to the magnetic field (i.e., when the magnetic field is perpendicular to the shock front).

For the results set out above to be valid the condition

$$R, l \gg \lambda = \lambda_0\frac{a_0 + U_k n_k}{a_{00} + U_{0k}n_{0k}}\times\left(1 + \frac{\gamma+1}{2}\frac{\lambda_0(a_{00} + U_{0k}n_{0k})}{\lambda_0}\int_0^t\frac{dt}{(a_0 + U_k n_k)\sqrt{\rho_0 a_0}L}\right)^{1/2} \qquad (2.10)$$

must be fulfilled, where the integral must be taken along the path of the front.

Moreover, for the condition $R_m \ll 1$ to be fulfilled, the condition

$$\sigma \ll \frac{c^2}{4\pi\lambda a_0}$$

must be fulfilled.

For $H \to 0$ or $\sigma \to 0$ the expressions (2.7), (2.8), (2.9) pass into the corresponding formulas of gasdynamics.

The author is grateful to B. I. Zaslavskii for useful advice and for discussing the paper.

REFERENCES

1. I. B. Keller, "Geometrical acoustics. I. The theory of weak shock waves," J. Appl. Phys., vol. 25, no. 8, 1954.

2. K. E. Gubkin, "The propagation of discontinuities in sound waves," PMM, 22, 4, 1958.

3. O. S. Ryzhov, "The damping of shock waves in inhomogeneous media," PMTF, no. 2, 1961.

4. Shih-i Pai, Magnetogasdynamics and Plasma Dynamics [Russian translation], Izd. Mir, 1964.

10 October 1965 Novosibirsk